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Ginzburg-Landau theory --- surface energy and vortices

Discussion the Ginzburg-Landau theory in three parts:

1. Presentation of the model and derivation of the penetration length and coherence length
2. Calculation of the surface energy and categorization of Type I and Type II superconductivity
3. Current-carrying states and phase coherence

Ginzburg-Landau Theory (summary)

G-L free energy: $\nabla G(\psi,\vec{A}) = G_S - G_N = \underbrace{\alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4}_{\text{CONDENSATION ENERGY}} + \underbrace{\frac{1}{2m^*}\left|\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e^*}{c}\vec{A}\right)\psi\right|^2}_{\text{KINETIC ENERGY}} + \underbrace{\frac{1}{8\pi}(B-H)^2}_{\text{MAGNETIC FIELD EXPULSION}}$

Order parameter : $\psi(\vec{r}) = |\psi(\vec{r})|\, e^{i\theta(\vec{r})}$ **phase**

Superelectron density : $|\psi(\vec{r})|^2 = n_s^*$

$$\frac{\hbar}{2m^*}\left|\vec{\nabla}(\psi)^2\right| + n_s^*\left(\frac{1}{2}m^*{v_s}^2\right)$$

Minimize $\int dV\, \Delta G(\psi,\vec{A})$ wrt $\psi,\vec{A} \Longrightarrow$

(1) $\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m^*}\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e^*}{c}\vec{A}\right)^2\psi = 0$

(2) $\vec{J}_s = \frac{c}{4\pi}\vec{\nabla}\times\vec{B} = \frac{e^*\hbar}{2m^*i}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) - \frac{(e^*)^2}{m^*c}|\psi|^2\vec{A}$

Interplay of the order parameter and magnetic fields

ISOLATED SAMPLE – no transport currents, ψ = real

(1) $\frac{\hbar^2}{4m}\nabla^2\psi = \left(\alpha + \frac{e^2}{me^2}A^2\right)\psi + \beta\psi^3$ ➡ $\xi(T) \equiv \left(\frac{\hbar^2}{4m|\alpha|}\right)^{1/2} = \frac{\hbar c}{2\sqrt{2}eH_c(T)\,\lambda(T)}$

coherence length

(2) $\nabla^2\vec{A} = \frac{8\pi e^2}{mc^2}\psi^2\vec{A} = \frac{4\pi e^2 n_s}{mc^2}\vec{A}$ ➡ $\lambda(T) = \left(\frac{mc^2}{4\pi e^2 n_s}\right)^{1/2}$

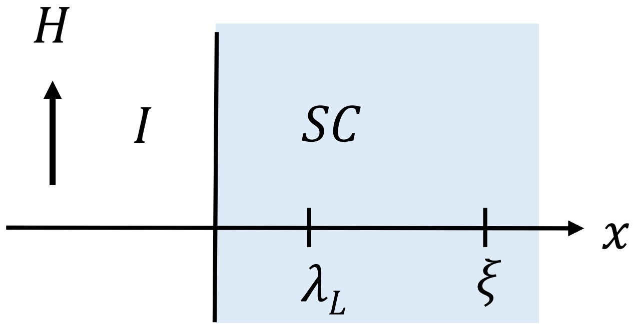
penetration length

$\kappa = \frac{\lambda(T)}{\xi(T)}$

G-L parameter

Variation of Order Parameter near a surface

$\xi > \lambda$ (Type I)



GL equations couple ψ and $\vec{A}$ so expect ψ to depend on field penetration

THICK SAMPLE $\quad d \gg \xi > \lambda$

Assume $\delta\Psi$ small → neglect terms in $\vec{\nabla}\psi$ and assume $\psi \sim \psi_\infty$ to get an estimate for $\vec{A}(x)$

GL2:

$$\vec{J}_s = \frac{n_s e^2}{mc}\overrightarrow{A}$$

$$\frac{d^2}{dx^2}\vec{A} = -\frac{4\pi n s e^2}{me^2}\vec{A} = -\left(\frac{1}{\lambda_L}\right)^2\vec{A}$$

$$A(x) = \lambda_L H e^{-\frac{x}{\lambda_L}}$$

GL1:

$$\frac{\hbar^2}{4m}\frac{d^2\Psi}{dx^2} = \left(\alpha + \frac{e^2}{mc^2}A^2(x)\right)\Psi(x) + \beta\Psi^3(x)$$

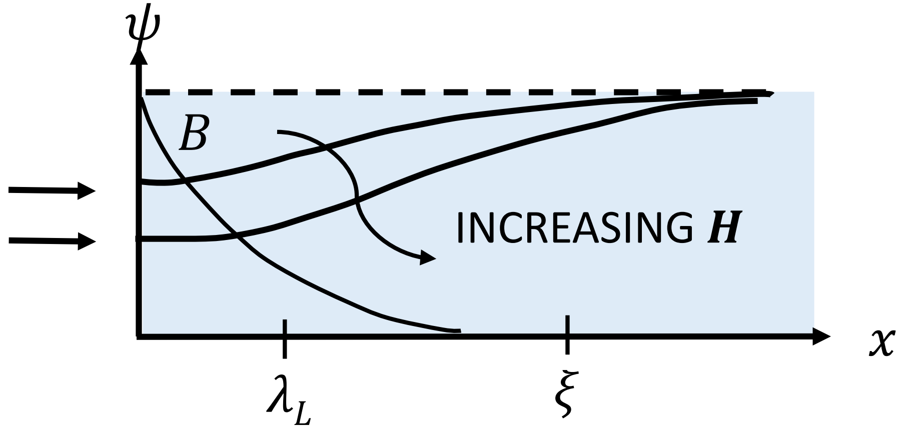
$$\Psi(x) = \Psi_\infty\left[1 - \frac{\kappa}{2\sqrt{2}\left(2-\kappa^2\right)}\left(\frac{H}{H_c}\right)^2\left(e^{-\frac{\sqrt{2}}{\xi}x} - \frac{\kappa}{\sqrt{2}}e^{-\frac{2}{\lambda}x}\right)\right]$$

B.C. $\quad \overrightarrow{A}\cdot\hat{n} = 0 \quad \Rightarrow \quad \frac{d\Psi}{dx} = 0$ at surface

$$\Psi(x)=\Psi_\infty\left[1-\frac{\kappa}{2\sqrt{2}\left(2-\kappa^2\right)}\left(\frac{H}{H_c}\right)^2\left(e^{-\frac{\sqrt{2}}{\xi}x}-\frac{\kappa}{\sqrt{2}}e^{-\frac{2}{\lambda}x}\right)\right]$$

Double exponential in ξ and λ

$$\frac{d\psi}{dx}=0$$



$$\frac{\Delta\Psi}{\Psi_0}(x=0)=\frac{\kappa}{4\sqrt{2}}\left(\frac{H}{H_c}\right)^2$$

Suppression of order parameter in field

Full self-consistent solution shows deviation of λ(x) from exponential as well

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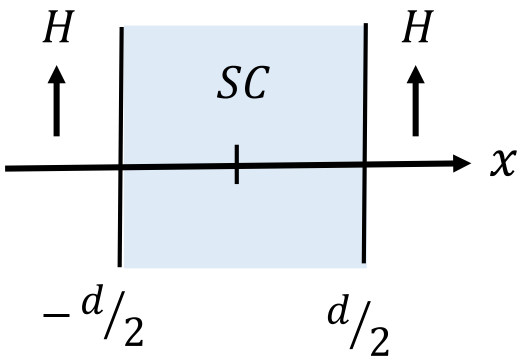
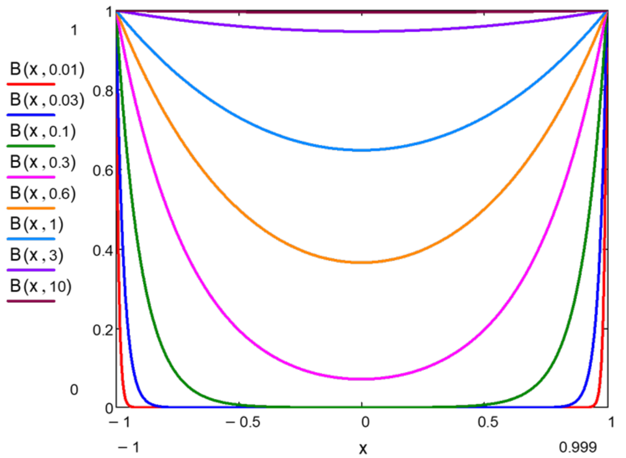
What happens in a thinner sample? a slab or a thin film





MEDIUM SAMPLE

$d \sim \lambda_L$ field can vary

$d \ll \xi$ uniform order parameter

Expect $\psi(x) = \psi < \psi_\infty$ (suppressed by field)

GL2: $\vec{J}_s = -\frac{e^2}{mc}|\Psi|^2\vec{A}$

$$\nabla^2\vec{A} = -\frac{8\pi e^2}{mc^2}|\Psi|^2\vec{A}$$

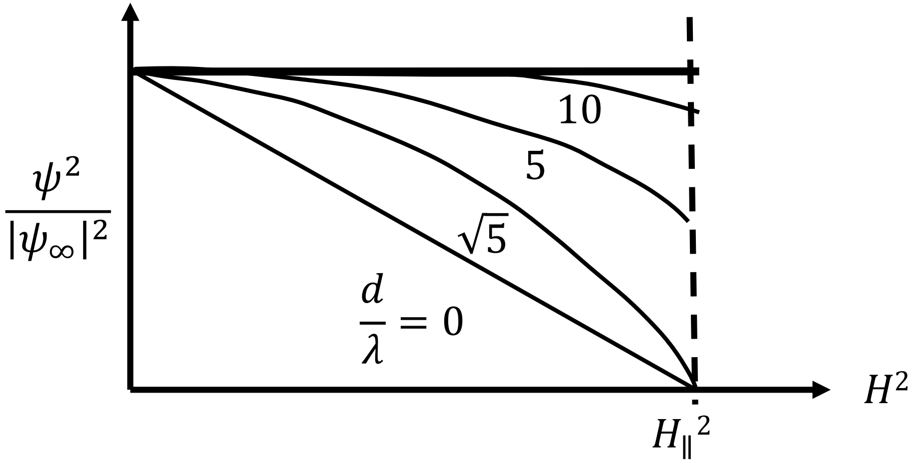
$$\therefore\ \lambda = \left(\frac{mc^2}{4\pi e^2 n_s}\right)^{\frac{1}{2}}\frac{|\psi_\infty|}{|\psi|} = \lambda_{eff} > \lambda_L$$

λ increased by the suppression of ψ

Calculated in Lecture 3 using the London equations

GL1: $$\frac{\hbar^2}{4m}\frac{d^2\Psi}{dx^2} = \left(\propto + \frac{e^2}{mc^2}A^2(x)\right)\Psi(x) + \beta\Psi^3(x) \quad \longleftarrow \quad A_z(x) = H\lambda_{eff}\frac{\sinh\left(\frac{x}{\lambda_{eff}}\right)}{\cosh\left(\frac{d}{2\lambda_{eff}}\right)}$$

$$\Psi(H) = \Psi_\infty\left[1 - \frac{1}{8}\left(\frac{H}{H_c}\right)^2 \frac{\sinh\left(\frac{d}{\lambda}\right) - \left(\frac{d}{\lambda}\right)}{\left(\frac{d}{\lambda}\right)\cosh^2\left(\frac{d}{\lambda}\right)}\right]$$



ψ suppresses $\Rightarrow$ λ grows and $|\Delta G|$ drops

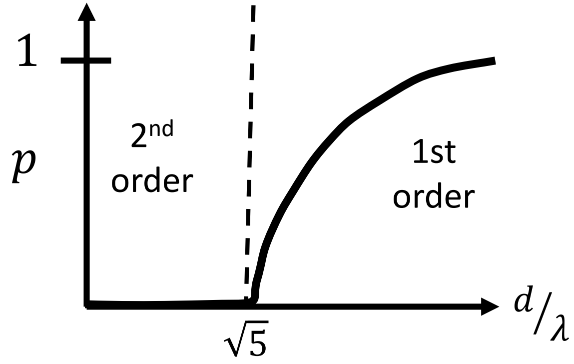
For $\frac{d}{\lambda} < \sqrt{5}$ (thin films), $\psi \to 0$ at the S → N transition

At some point $\Delta G \to 0 \Rightarrow S \to N$ transition
This defines a critical field $H_{\parallel}$ (parallel)

For $\frac{d}{\lambda} > \sqrt{5}$ (thick films), Ψ partially suppressed at the transition

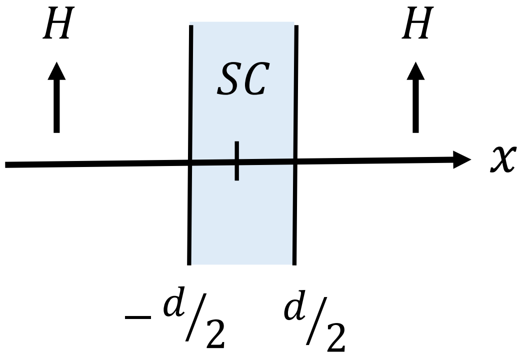
$$H_{\parallel} = H_c\left[\frac{p^2(2-p^2)}{1 - \left(\frac{2\lambda}{p\,d}\right)\tanh\left(\frac{d\,p}{2\lambda}\right)}\right]^{1/2}$$

$p = \frac{\psi}{\psi_\infty}$ at the phase transition



THIN FILMS

d << ξ and d << λ_L



From before: p = 0

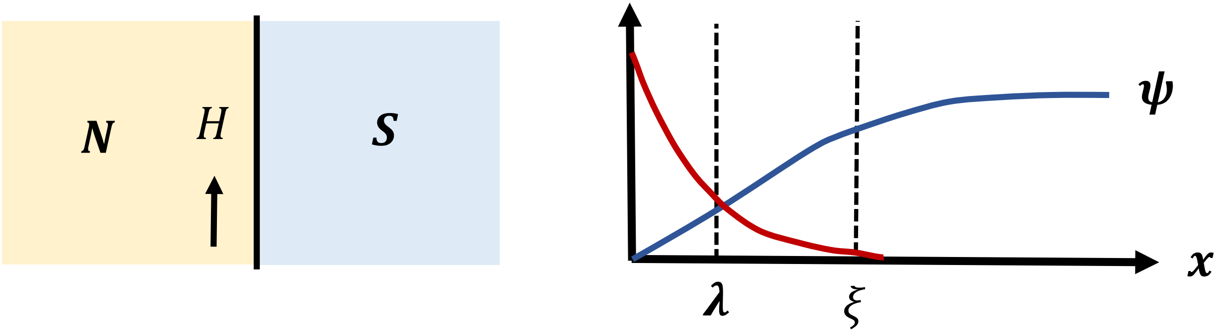
$$\Psi(H) = \Psi_\infty \left[1 - \frac{d^2}{24\lambda^2}\left(\frac{H}{H_c}\right)^2\right]^{1/2} = \Psi_\infty \left[1 - \left(\frac{H_{\parallel}}{H_c}\right)^2\right]^{1/2}$$

$\psi \to 0$ when $H = H_{\parallel} = 2\sqrt{6}\,\dfrac{\lambda}{d} H_c >> H_c$

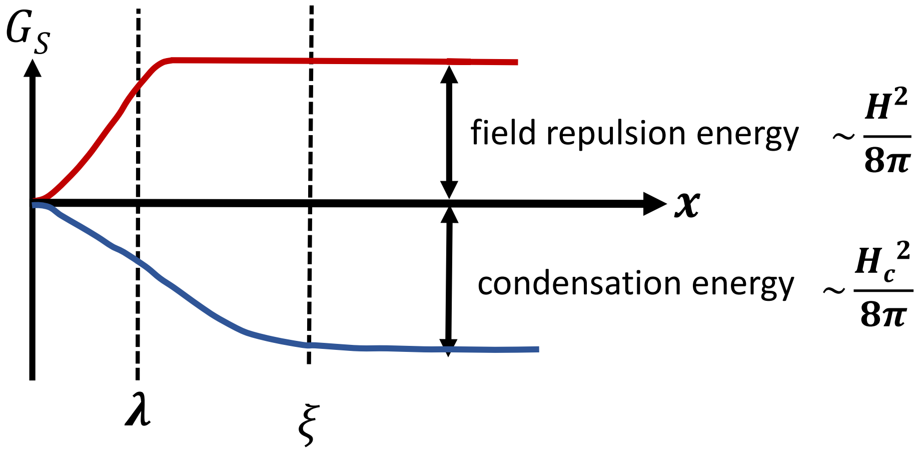
Why? $H_{\parallel} > H_c$ because the field does not have to fully exclude the field from bulk to maintain SC

Relevant for Type II $\Rightarrow$ $H_{c2} \gg H_c$

Surface Energy



Calculate contributions to Gibbs free energy at interface



Allow field penetration $\sigma \sim -\lambda \left(\frac{H^2}{8\pi}\right)$ **lowers** SC energy

Lose SC region $\sigma \sim +\ \xi \left(\frac{H_c{}^2}{8\pi}\right)$ **raises** SC energy

$$\sigma = \frac{H_c{}^2}{8\pi}\xi - \frac{H^2}{8\pi}\lambda$$

$= \frac{H_c{}^2}{8\pi}(\xi - \lambda)$, since H = H_c at the phase boundary

$$\equiv \frac{H_c{}^2}{8\pi}\delta$$

Bulk: $\Delta G = GS - GN = \frac{1}{8\pi}(H^2 - H_c{}^2) < 0 \quad \Rightarrow \ SC$

Surface: $\sigma = \int \Delta G \, dV$ (integrate over surface region)

$\kappa < 1\ (\lambda < \xi)$

$\sigma > 0$

N-S boundaries cost energy
Resists formation of vortices

$\kappa > 1\ (\lambda > \xi)$

$\sigma < 0$

N-S boundaries lower the system energy
Encourages formation of vortices

Ginzburg – London treatment

$$\sigma = \int_{-\infty}^{\infty} dx\,(g_S - g_N) \;=\; \int_{-\infty}^{\infty} dx \left\{ \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{1}{4m}\left|\left(\frac{\hbar}{i}\vec{\nabla} - \frac{2e\vec{A}}{c}\right)\psi\right|^2 + \frac{(B-H_c)^2}{8\pi} \right\}$$

g = energy per length

1st GL equation: $\alpha\psi + \beta|\psi|^2\psi + \frac{1}{4m}\left(\frac{\hbar}{i}\vec{\nabla} - \frac{2e}{c}\vec{A}\right)^2\psi = 0$

Multiply by ψ^* and integrate $\Longrightarrow$

$$\int_{-\infty}^{\infty} dx \left\{ \alpha|\psi|^2 + \beta|\psi|^4 + \frac{1}{4\pi}\left|\left(\frac{\hbar\vec{\nabla}}{i} - \frac{2e\vec{A}}{c}\right)\psi\right|^2 \right\} = 0$$

Subtract from σ expression $\Longrightarrow$

$$\sigma = \int_{-\infty}^{\infty} dx \left\{ \frac{(B-Hc)^2}{8\pi} - \frac{1}{2}\beta|\psi|^4 \right\} = \frac{{H_c}^2}{8\pi}\int_{-\infty}^{\infty} dx \sum \left(\frac{B-Hc}{H_c}\right)^2 - \frac{4\pi}{{H_c}^2}\beta|\psi|^4 = \frac{{H_c}^2}{8\pi}\int_{-\infty}^{\infty} dx \left\{ \left(\frac{B-Hc}{H_c}\right)^2 - \left(\frac{\psi}{\psi_\infty}\right)^4 \right\}$$

using $|\psi_\infty|^2 = -\frac{\alpha}{\beta}$ and $\frac{{H_c}^2}{8\pi} = \frac{\alpha^2}{\beta} \equiv \frac{{H_c}^2}{8\pi}\delta$

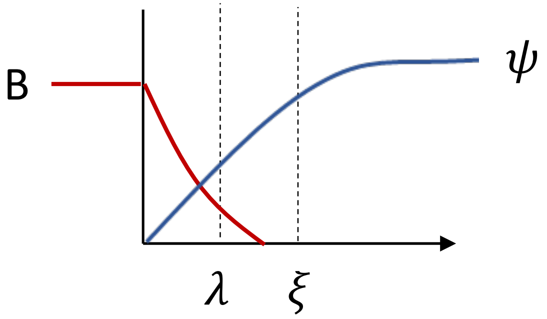
$$\delta = \int_{-\infty}^{\infty} dx \left\{ \left(\frac{B - Hc}{H_c} \right)^2 - \left(\frac{\Psi}{\Psi_\infty} \right)^4 \right\}$$

↑ field penetration $\sim\lambda$

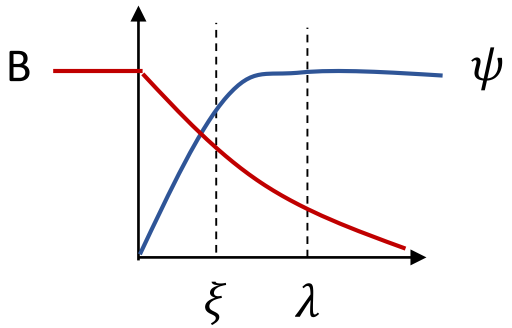
↑ order parameter variation $\sim\xi$

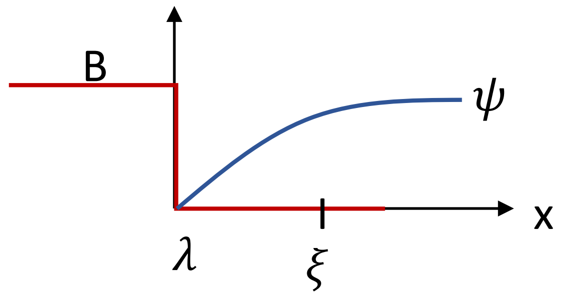
Solve $\psi(x)$, $H(x)$ from GL $\Longrightarrow$ calculate δ, σ

$\kappa < 1 \quad (\lambda < \xi)$



$\kappa > 1 \quad (\lambda > \xi)$





<u>Limits:</u>

① κ small $(\lambda \to 0)$

Assume no field penetration:

$$\xi^2 \frac{d^2\psi}{dx^2} + \psi - \psi^3 = 0$$

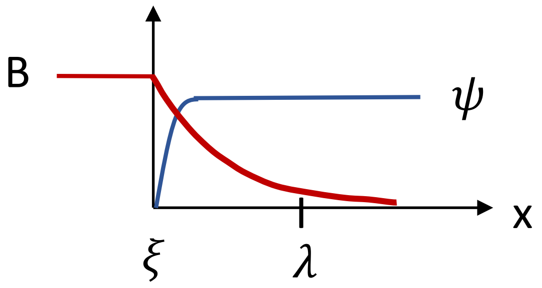
$$\psi(x) = \psi_0 \tanh\left(\frac{x}{\sqrt{2}\xi}\right), \text{ for } x > 0$$

$$\delta = \int_0^\infty dx \left[1 - \tanh^4\left(\frac{x}{\sqrt{2}\xi} \right) \right] = \frac{4}{3}\sqrt{2}\,\xi = 1.89\,\xi$$

$$\boxed{\sigma = 1.89\xi \left(\frac{H_c^2}{8\pi} \right)}$$

Self-consistently, variation of $\psi \Rightarrow \lambda_{eff} = \sqrt{\lambda\,\xi}$

② κ large ($\xi \to 0$)



Assume $B(x) = Hc\, e^{-x/\lambda}$

$$\delta = \int_0^\infty dx \left[\left(e^{-x/\lambda} - 1\right)^2 - 1\right] = -1.5\lambda$$

$$\sigma = -1.5\lambda\left(\frac{{H_c}^2}{8\pi}\right)$$

Better, consider ψ variation which cannot be neglected due to $\vec{\nabla}\psi$ terms. Find:

$$\boxed{\sigma = -\frac{8}{3}(\sqrt{2} - 1)\lambda\,\frac{{H_c}^2}{8\pi} = -1.10\,\lambda\left(\frac{{H_c}^2}{8\pi}\right)}$$

③ $\kappa \sim 1$ (crossover region)

$$\psi(x) \sim \psi_\infty\left(1 - e^{-\frac{x^2}{2\xi^2}}\right)$$

$$B(x) \sim \psi^2$$

$$\delta = 0 \Rightarrow \kappa = \frac{1}{\sqrt{2}}$$

This will determine whether a material is

Type I $\quad \kappa < \frac{1}{\sqrt{2}}$

Type II $\quad \kappa > \frac{1}{\sqrt{2}}$

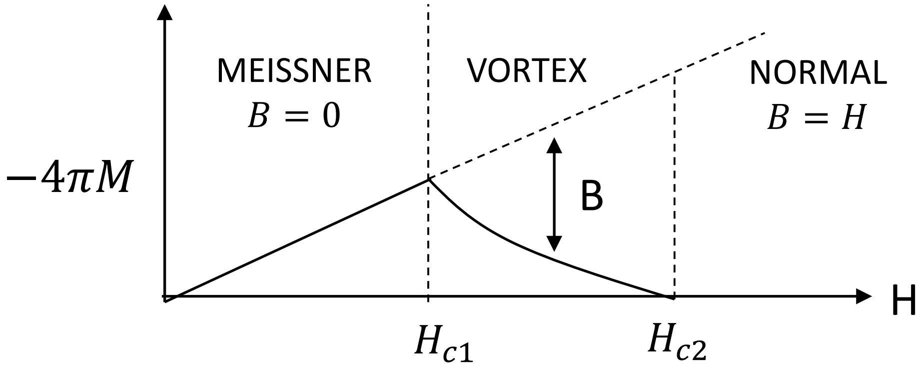
But phase coherence plays a key role in the nature of vortex state

<u>Type II Superconductivity</u> $H > "H_c" \Longrightarrow$ field penetrates in discrete vortices

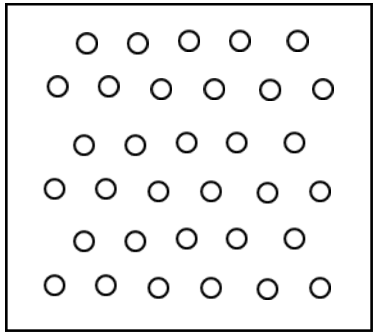
VORTEX STATE

(1) $\kappa = \frac{\lambda}{\xi} > \frac{1}{\sqrt{2}}$ $\Longrightarrow$ negative surface energy $\Longrightarrow$ maximize N-S interface area

(2) Fluxoid quantization $\Longrightarrow$ smallest flux unit $= \Phi_o$



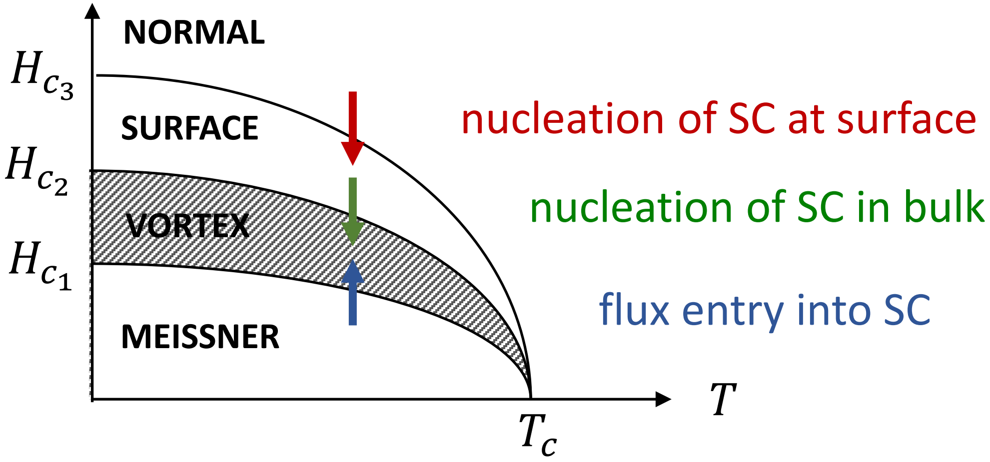
Vortex density: $n = \frac{B}{\Phi_o}$



$$a_\Delta = \left(\frac{4}{3}\right)^{\frac{1}{4}} \left(\frac{\Phi_0}{B}\right)^{\frac{1}{2}} = 1.075 \left(\frac{\Phi_0}{B}\right)^{\frac{1}{2}}$$

$$a_\square = \left(\frac{\Phi}{B}\right)^{\frac{1}{2}} = 1.000 \left(\frac{\Phi_0}{B}\right)^{\frac{1}{2}}$$

Triangle lattice more stable (barely)



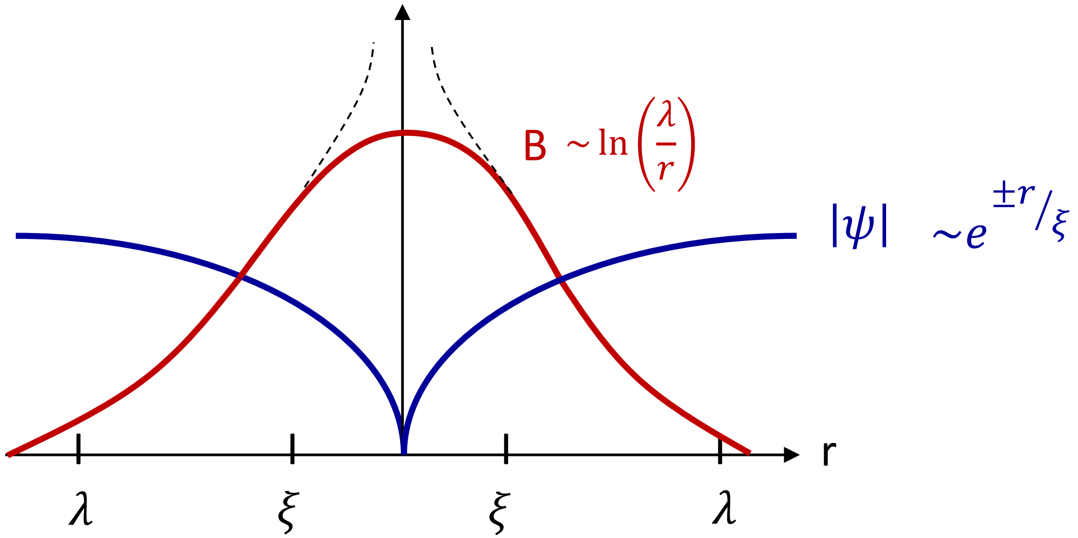
VORTEX STRUCTURE

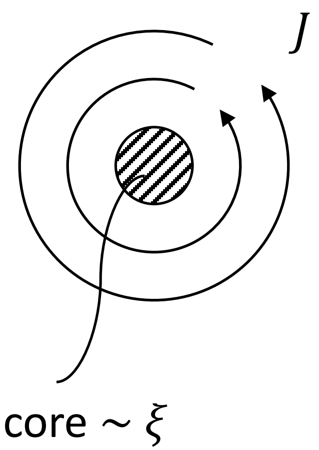
Abrikosov vortex

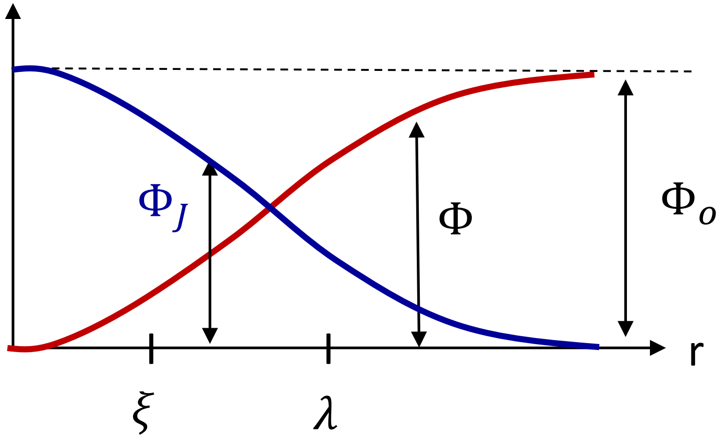
(1) B varies over λ

(2) $|\psi|^2$ varies over ξ_{GL} --- defines "core" of the vortex

(3) B flattens off near core because $|\psi|^2 \to 0$







We will show next time that the quantity that is quantized is the "fluxoid" (not the magnetic flux):

$$\Phi' = \frac{mc}{n_S e^2} \oint \vec{J_S} \cdot \vec{dl} + \oint \vec{A} \cdot \vec{dl} = \Phi_J + \Phi$$

Near core, dominated by $\Phi_J = \frac{mc}{n_S e^2} \oint \vec{J_S} \cdot \vec{dl}$

Far away, Φ' dominated by $\Phi = \oint \vec{A} \cdot \vec{dl}$

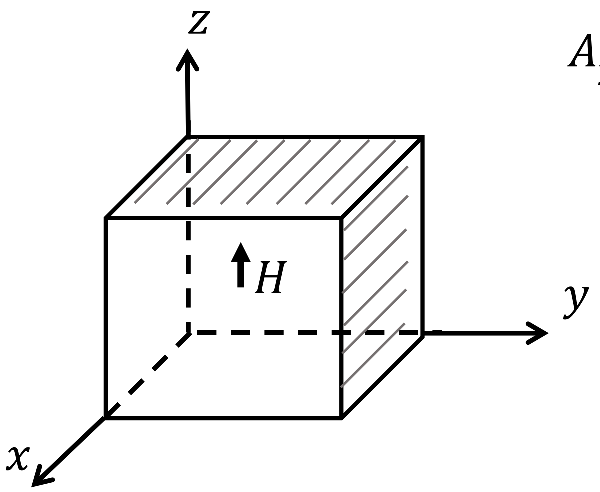
Quantization ⇨ $\Phi' = \Phi_0$ at all r

Nucleation of SC in the bulk

When does SC start as H is lowered?

Use linearized GL equation:

$$\frac{1}{2m^*}\left(\frac{\hbar}{i}\vec{\nabla}-\frac{e^*}{c}\bar{A}_{ext}\right)^2\psi+\propto\psi+\ \beta|\psi|^2\psi=0$$



$A_y = Hx$ (gauge choice)

$\vec{H} = H\hat{z}$

- Neglect cubic term – valid for $|\psi| \ll |\psi_N|$ which will occur at high fields as SC onsets
- $\vec{A} = \vec{A}_{ext}$ since corrections are of order $|\psi|^2$ this decouples the two GL equations into one for ψ:

$$\left(\frac{\vec{\nabla}}{i}-\frac{2\pi H}{\Phi_0}x\,\hat{y}\right)^2\psi=-\frac{2m^*\alpha}{\hbar^2}\psi=\frac{1}{\xi^2}\psi$$

$$\left[-\nabla^2+i\frac{4\pi H}{\Phi_0}x\frac{\partial}{\partial y}+\left(\frac{2\pi H}{\Phi_0^2}\right)x^2\right]\psi=\frac{1}{\xi^2}\psi$$

Assume $\psi(x,y,z)=f(x)e^{ik_y y}e^{ik_z z}$ ⇨ $-\frac{d^2f}{dx^2}+\left(\frac{2\pi H}{\Phi_0}\right)^2\left(x-\frac{k_y\Phi_0}{2\pi h}\right)^2 f=\left(\frac{1}{\xi^2}-k_z^2\right)f$

$$-\frac{d^2 f}{dx^2}+\left(\frac{2\pi H}{\Phi_0}\right)^2\left(x-\frac{k_y\Phi_0}{2\pi h}\right)^2 f=\left(\frac{1}{\xi^2}-k_z^2\right)f$$

Harmonic oscillator Schrödinger equation $x\left(-\frac{\hbar^2}{2m}\right)$ with potential:

$$-\frac{\hbar^2}{2m^*}\frac{\partial^2}{\partial x^2}\psi+\frac{1}{2}k(x-x_0)^2\psi=E\psi$$

Solutions are quantized state called Landau levels:

$$E_n=\left(n+\frac{1}{2}\right)\hbar\omega=\left(n+\frac{1}{2}\right)\hbar\left(\frac{2eH}{m^*c}\right)=\frac{\hbar^2}{2m^*}\left(\frac{1}{\xi^2}-k_z^2\right)$$

$$H_n=\frac{\Phi_0}{2\pi(2n+1)}\left(\frac{1}{\xi^2}-k_z^2\right)$$

Choose greatest $H\equiv H_{c2}$ for $n=0$ and $k_z=0$:
(first SC solution at highest H)

$$H_{c2}=\frac{\Phi_0}{2_{\pi}\xi^2}=\frac{4_{\pi}\lambda^2 H_c^2}{\Phi_0}=\sqrt{2}\,\kappa\,H_c$$ "upper critical field"

using $\xi=\frac{\hbar c}{2\sqrt{2}eH_c\lambda}$ and $\kappa=\frac{\lambda}{\xi}$

$$u=\frac{1}{2}\underbrace{\left[\frac{1}{m^*}\left(\frac{2\pi\hbar H}{\Phi_0}\right)^2\right]}_{k}(x-x_0)^2,$$

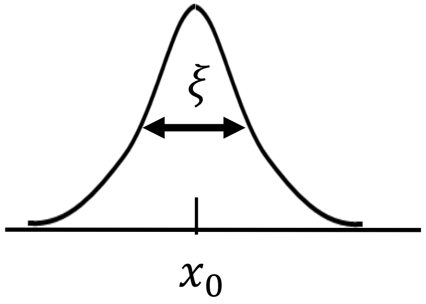
k "spring constant"

$$x_0=\frac{k_y\,\Phi_0}{2\pi H}$$ "equilibrium position"

$$\omega=\sqrt{\frac{k}{m^*}}=\frac{2eH}{m^*c}$$ "oscillation frequency"

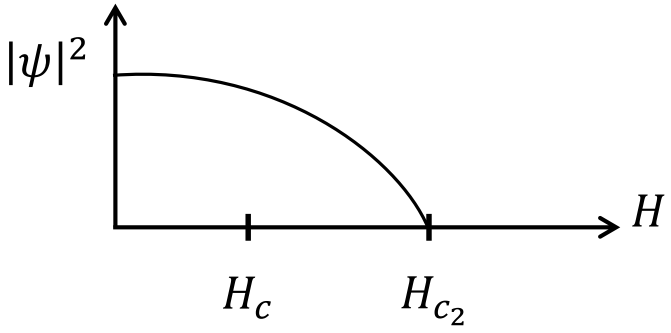
Wavefunction:

$$\psi(x)\sim e^{-\frac{(x-x_0)^2}{2\xi^2}}$$



nucleation of SC at $x=x_0$

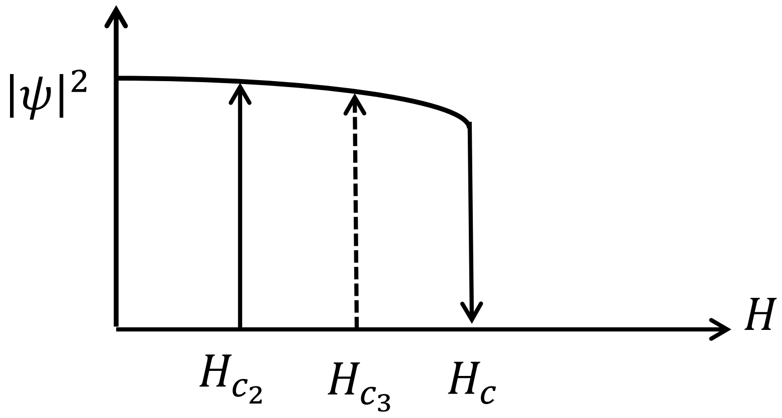
For $\kappa > \frac{1}{\sqrt{2}}$ $H_{c_2} > H_c$ Type II SC



2^{nd} order transition

SC at fields above H_c due to flux penetration

For $\kappa < \frac{1}{\sqrt{2}}$ $H_{c_2} < H_c$



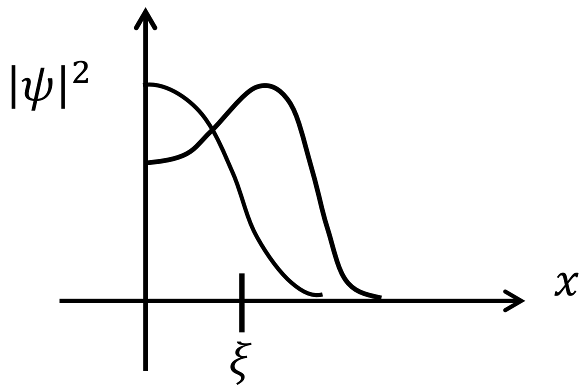
1^{st} order transition

supercooling before H_c before SC onsets

Has been observed --- can be used to measure κ but is usually destroyed by surface roughness
Actually, will only supercool to H_{c_3} (where surface SC onsets)

Surface SC

This neglects fall off of $|\psi|^2$ at surface (over ξ)



Energy lower at surface $\Longrightarrow$ SC forms at higher value of H

$$H_{C_3} = 1.695 H_{c_2} = (2.40\,\kappa)\,H_c$$

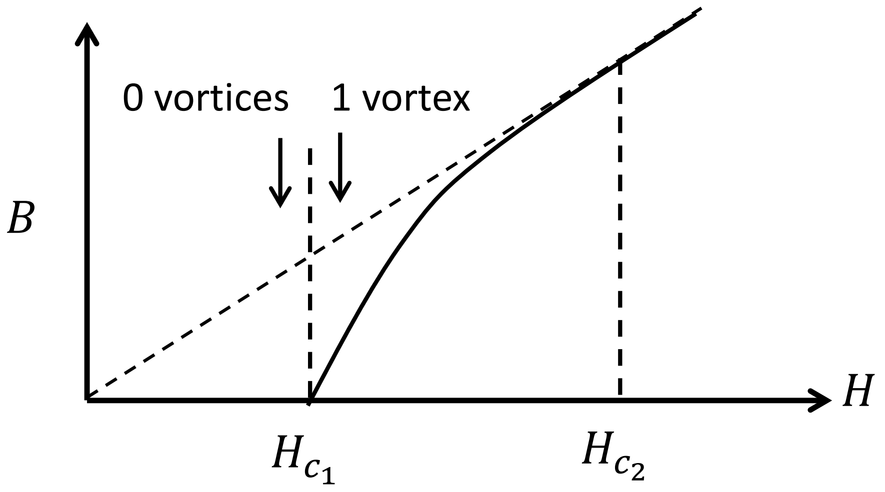
Supports layer of SC of with width ξ

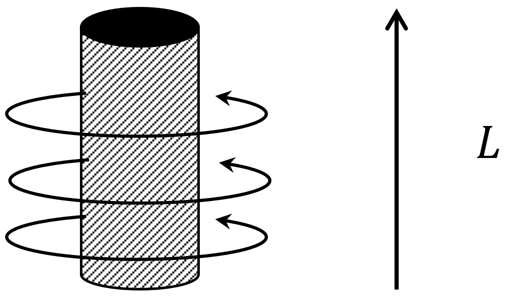
Can get SC's with $H_{c_2} < H_c < H_{c_3} \Longrightarrow$ 1st order but no supercooling

<u>Vortex Nucleation</u>: when do vortices enter as H raises?

Define critical field H_{c_1}

At H_{c_1} $\quad G_S^{Meissner} = G_s^{vortex}$





Let ε_ℓ = line energy of vortex/length

Tradeoff vortex energy vs. field energy (to allow B to penetrate)

$$\varepsilon_\ell L = \frac{1}{4_\pi}\int \overline{B\cdot}\,\overline{H} \propto \upsilon \quad = \frac{H_{c_1}}{4_\pi}\int B\ dV = \frac{H_{c_1}}{4_\pi}\left(\int B\ dV\right)L \quad = \frac{H_{c_1}}{4_\pi}\ \ \Phi_0\ L$$

$$\therefore H_{c_1} = \frac{4_\pi \varepsilon_\ell}{\Phi_0}$$ "lower critical field"

ε_ℓ? Must solve GL to get vortex slope: $\psi(r), A(r)$

Calculate line energy (field energy $+KE$

Guess: $\varepsilon_\ell \sim \left(\frac{H_c^2}{8_\pi}\right)\lambda^2 = \left(\frac{H_c^2}{8_\pi}\right)\xi^2 \sim \left(\frac{H_c}{8_\pi}\right)^2 (\lambda^2 - \xi^2)$

field energy

condensation energy

Solutions $\left(\kappa \gg \frac{1}{\sqrt{2}}\right)$ use full GL

$$\psi(r) \sim |\psi_\infty| \tanh \frac{r}{\xi}$$

$$B(r) = \left(\frac{\Phi_0}{2_\pi \lambda^2}\right) K_0\left(\frac{r}{\lambda}\right) = H_{c_2} K_0\left(\frac{r}{\lambda}\right) = \begin{cases} \frac{\Phi_0}{2_\pi \lambda^2}\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{\lambda}{r}\right)^{1/2} e^{-r/\lambda} & r \gg \lambda \quad \text{(long range)} \\ \frac{\Phi_0}{2_\pi \lambda^2}\left[\ell n\left(\frac{\lambda}{r} + 0.12\right)\right] & \xi \ll r \ll \lambda \quad \text{(short range)} \end{cases}$$

zero – order Hankel function

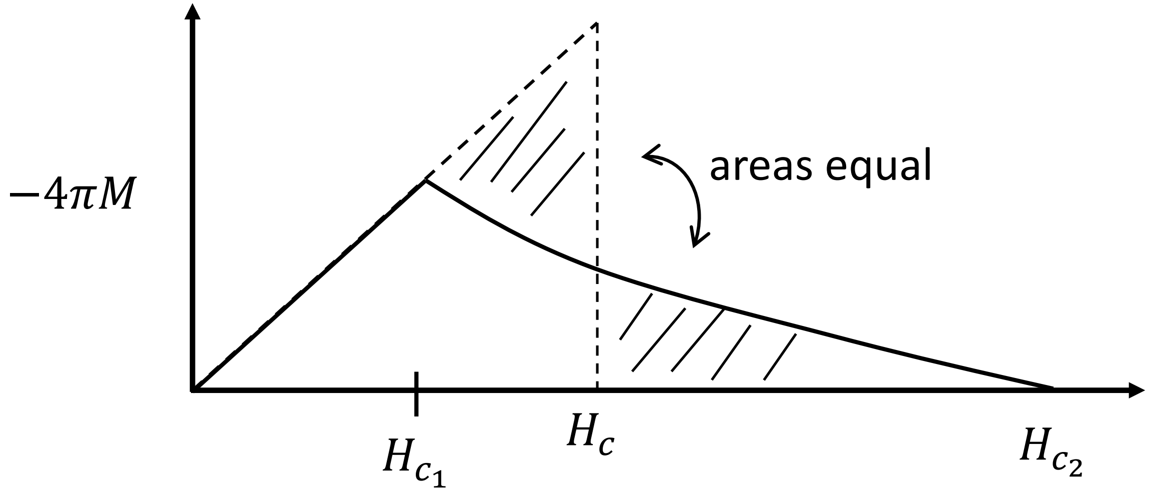
$B(r)$ does not diverge in core – flattens off as $|\psi|^2 \to 0$ near center

Put back in: $\varepsilon_\ell = \left(\frac{\Phi_0}{4_\pi \lambda}\right)^2 \ell n\, \kappa = \left(\frac{H_c^2}{8_\pi}\right) 4_\pi \xi^2 \ell n\, \kappa$

$$H_{c_1} = \frac{4_\pi}{\Phi_0}\varepsilon_1 = \frac{\Phi_0}{4_\pi \lambda^2}\ell n\, \kappa = H_c \frac{\ell n\, \kappa}{\sqrt{2}\,\kappa}$$

$$H_{C_2} = \sqrt{2}\,\kappa\, H_c$$

$$H_c = \frac{1}{\sqrt{\ell n\, \kappa}}\left(H_{c_1} H_{c_2}\right)^{1/2}$$



$$H_{c_1} \sim \frac{\Phi_0}{\lambda^2}$$

$$H_{c_2} \sim \frac{\Phi_0}{\xi^2}$$